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Critical Value of the Quantum Ising Model on Star-Like Graphs

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Abstract We present a rigorous determination of the critical value of the ground-state quantum Ising model in a transverse field, on a class of planar graphs which we call *star-like*. These include the junction of several copies of $\mathbb{Z}$ at a single point. Our approach is to use the graphical, or FK-, representation of the model, and the probabilistic and geometric tools associated with it.

Keywords Ising model · Random-cluster model · Critical value

1 Introduction

The Hamiltonian of the quantum Ising model with transverse field on a finite graph $G = (V, E)$ is the operator

$$H = -\frac{1}{2}\lambda \sum_{e=xy\in E} \sigma_x^{(3)}\sigma_y^{(3)} - \delta \sum_{y\in V} \sigma_x^{(1)} \tag{1}$$

on the Hilbert space $\mathcal{H} = \bigotimes_{x\in V} \mathbb{C}^2$. Here the Pauli spin-1/2 matrices

$$\sigma_x^{(3)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma_x^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2}$$

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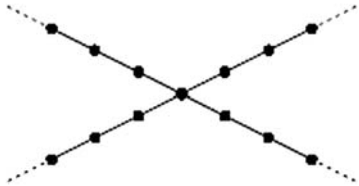

Fig. 1 The star graph has a central vertex of degree $k \geq 3$ and k infinite arms, on which each vertex has degree 2. In this illustration, $k = 4$

and we use as basis for each copy of $\mathbb{C}^2$ in $\mathcal{H}$ the vectors $|+\rangle_x = \binom{1}{0}$ and $|-\rangle_x = \binom{0}{1}$; also, $\lambda, \delta > 0$ are the spin-coupling and external-field intensities, respectively. Let $\beta \geq 0$ denote the inverse temperature, and define the positive temperature states

$$\rho_{G,\beta}(Q) = \frac{1}{Z_G(\beta)} \mathrm{tr}(e^{-\beta H} Q), \tag{3}$$

where $Z_G(\beta) = \mathrm{tr}(e^{-\beta H})$ and $Q \in \mathbb{C}^{2\times 2}$. Also define the *ground state* to be the limit ρ_G of $\rho_{G,\beta}$ as $\beta \to \infty$. If G_n is an increasing sequence of graphs tending to an infinite graph S, then we may also speak of infinite-volume limits $\rho_{S,\beta} = \lim_{n\to\infty} \rho_{G_n,\beta}$ and $\rho_S = \lim_{n\to\infty} \rho_{G_n}$. The existence of these limits is discussed in [3].

In this article we will use the FK- or random-cluster representation of the ground state, see for example [16] and references therein. Details will be provided in the next section, but roughly speaking the FK-representation may be considered as a limit of “discrete time” random-cluster models on $S \times (\varepsilon\mathbb{Z})$ as $\varepsilon \downarrow 0$. This is related to the well-known mapping of the quantum Ising model onto the classical Ising model in one dimension higher [21], and the FK-representation of that model [12]. The relevance of this representation is that it relates the occurrence of *long range order* in the ground state to the existence of infinite percolation paths in $S \times \mathbb{R}$; here we say that the model exhibits long range order if for all x, the correlation function

$$G(x, y) = \rho_S(\sigma_x^{(3)} \sigma_y^{(3)}) \tag{4}$$

is bounded below by a positive function of x. There is a critical value of the ratio λ/δ above which the model exhibits long range order, and below which it does not.

The main result of this article is a rigorous determination of the critical ratio for a certain class of planar graphs S (see Definition 1). This extends the calculation for the graph $S = \mathbb{Z}$, to, amongst other graphs, the *star graph*, which is the junction of several copies of $\mathbb{Z}$ at a single point. See Fig. 1. A special case of our main result (Theorem 2) is therefore the following.

Theorem 1 *The critical ratio for the ground state quantum Ising model on the star graph is* $\lambda/\delta = 2$.

In other words the critical ratio is the same for the star as for $\mathbb{Z}$; this is to be expected since the star is only *locally* different from $\mathbb{Z}$. We emphasise, however, that the class of graphs for which we prove this result contains many more graphs than just the star.

The quantum Ising model on $\mathbb{Z}$ is exactly solvable and has been thoroughly studied. In for example [20] the critical ratio $\lambda/\delta = 2$ is computed for this model, and it is also proved that correlations decay exponentially below the critical point; see also [21] and references therein. The latter fact will be used extensively in this paper. Typically, the proofs have relied on matrix methods and techniques such as Jordan–Wigner transformation. Recently, in [7], sharpness of the phase transition, and hence exponential decay of correlations below the

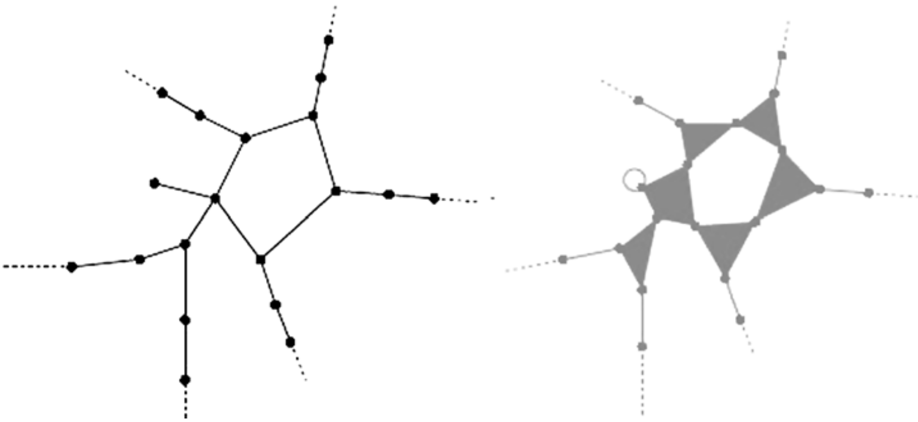

Fig. 2 A star-like graph G (*left*) and its line-hypergraph H (*right*). Any vertex of degree ≥ 3 in G is associated with a "polygonal" hyperedge in H

critical point, was established rigorously for $S = \mathbb{Z}^d$ with any $d \geq 1$, using graphical methods similar to the corresponding proof [2] for the classical Ising model. Combining this result with duality arguments analogous to the classical two-dimensional random-cluster model [12], this gives another proof that the critical ratio is $\lambda/\delta = 2$ for this model, using only tools from stochastic geometry (see [7] for details). One aim of this paper is to extend the graphical methods, and show how they can be applied to a wider range of structures than just $\mathbb{Z}$. The Ising model on the star-graph has also recently arisen in the study of boundary effects in the two-dimensional classical Ising model, see for example [18, 19]. Similar geometries have also arisen in different problems in quantum theory, such as transport properties of quantum wire systems, see [10, 15, 17].

2 Background and Notation

In this article we will let $G = (V, E)$ be a *star-like graph*:

Definition 1 A star-like graph is a countably infinite connected planar graph, in which all vertices have finite degree and only finitely many vertices have degree larger than two.

Such a graph is illustrated in Fig. 2; note that the graph of Theorem 1 is an example in which exactly one vertex has degree at least three.

Fix a planar embedding $\mathbb{G}$ of G, and denote $\mathbb{X} = \mathbb{G} \times \mathbb{R}$; also let $X = G \times \mathbb{R} := (V \times \mathbb{R}, E \times \mathbb{R})$. We will sometimes use X and $\mathbb{X}$ interchangably. Let $\mathcal{O}$ be a fixed but arbitrary vertex of G of degree two or more, which we think of as the origin.

Recall that a *hypergraph* is a set W together with a collection F of subsets of W, called *edges*; a graph is a hypergraph in which all edges contain two elements. In our analysis we will use a suitably defined hypergraph "dual" of $\mathbb{X}$: let $H = (W, F)$ be the "*line-hypergraph*" of G, where $W = E$ and the set $\{e_1, \dots, e_n\} \subseteq E = W$ is in F if and only if $e_1, \dots, e_n$ are all the edges adjacent to some particular vertex of G. Note that only finitely many edges of H have size larger than two. There is a natural planar embedding of $\mathbb{H}$ defined via the embedding $\mathbb{G}$, in which an edge of size more than two is represented as a polygon. See Fig. 2. Let $Y = H \times \mathbb{R}$ and $\mathbb{Y} = \mathbb{H} \times \mathbb{R}$.

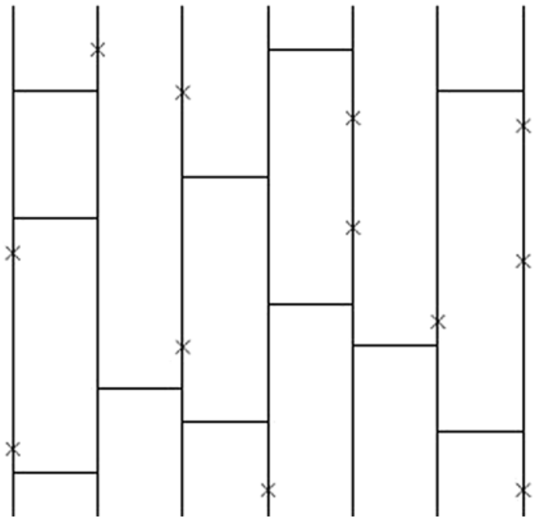

Fig. 3 A configuration ω on $\mathbb{Z} \times \mathbb{R}$. Bridges are represented as *horizontal line* segments, and deaths as *crosses*

Our configuration space Ω will be the set of pairs $\omega = (B, D)$ where $B \subseteq E \times \mathbb{R}$ and $D \subseteq V \times \mathbb{R}$ are *locally finite*, which is to say that $B \cap (\{e\} \times [-n, n])$ and $D \cap (\{v\} \times [-n, n])$ are finite sets for all $e \in E$, $v \in V$ and $n \in \mathbb{N}$. We think of B as a set of *bridges* and D as a set of *deaths* or cuts. There is a natural embedding of any $\omega \in \Omega$ into $\mathbb{X}$, where deaths are represented as missing points and bridges as "horizontal" lines connecting two "vertical" lines. See Fig. 3 for an illustration of this when $G = \mathbb{Z}$. Often we will identify $\omega \in \Omega$ with its embedding, $\omega \equiv (\mathbb{X} \setminus D) \cup B$.

Denote by $d(\cdot, \cdot)$ the graph distance in G, and let $\Lambda_n \subseteq \mathbb{X}$ denote the set of points (v, t) and (e, t) where $v \in V$ has $d(v, \mathcal{O}) \leq n$, $e \in E$ has at least one endpoint at distance at most n from $\mathcal{O}$, and $|t| \leq n$. For each $\omega \in \Omega$, we will employ two *restricted embeddings* ω_n^1 and ω_n^0 of ω into $\mathbb{X}$, one "wired" and one "free". The free embedding ω_n^0 is simply the intersection of (the natural embedding of) ω with Λ_n. The wired embedding ω_n^1 is defined by

$$\omega_n^1 = \omega_n^0 \cup \{(v, t) \in V \times \mathbb{R} : d(v, \mathcal{O}) = n + 1, |t| \leq n\} \cup \{(e, t) \in \Lambda_n : t = \pm n\}, \quad (5)$$

where we have identified $v \in V$ and $e \in E$ with their embeddings in $\mathbb{G}$. In words, ω_n^1 is obtained by tying together the top and bottom of ω_n^0, as well as all bridges portruding from its "sides". We let the functions $k_n^0, k_n^1 : \Omega \to \mathbb{N}$ count the number of connected components of ω_n^0 and ω_n^1, respectively.

Equip Ω with the Skorokhod topology and the associated σ-algebra; the details of their definitions are not immediately important, but may be found in [5] or [6]. Fix $\lambda, \delta > 0$ and let $\mu = \mu_{\lambda,\delta}$ be the probability measure on Ω governed by a collection of independent Poisson processes B_e on $\{e\} \times \mathbb{R}$, for $e \in E$, and D_v on $\{v\} \times \mathbb{R}$, for $v \in V$. Here each B_e has intensity λ, each D_v has intensity δ, and $B = \bigcup_{e \in E} B_e$, $D = \bigcup_{v \in V} D_v$. This μ is the space-time (or "continuum") percolation measure of [13]. We may now define the random-cluster probability measures.

Definition 2 The random-cluster measure Φ_n^b on Λ_n with parameters $\lambda, \delta, q > 0$ and boundary condition $b \in \{0, 1\}$ is the probability measure on Ω given by

$$\frac{d\Phi_n^b}{d\mu}(\omega) \propto q^{k_n^b(\omega)}, \quad \omega \in \Omega. \quad (6)$$

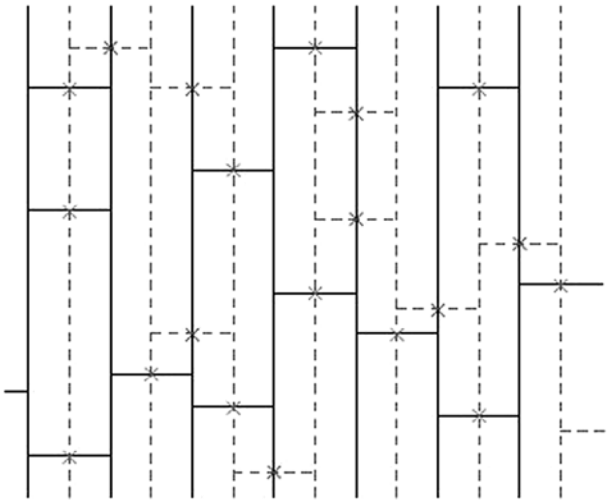

Fig. 4 Part of a configuration ω (*solid*) and its dual ω_{d} (*dashed* with *grey crosses*) in the special case when $G=\mathbb{Z}$

In the next result, let

$$\theta^b = \Phi^b((\mathcal{O},0) \text{ lies in an unbounded component}). \tag{7}$$

The following basic facts may be proved in a conventional manner, as in [12, Theorem 5.5]; details for this particular model may be found in [6].

Proposition 1 *Let $q \geq 1$. The weak limits $\Phi^b := \lim_{n\to\infty} \Phi_n^b$ exist, and enjoy a* phase transition *in the sense that there is $\rho_c = \rho_c(q) \in (0,\infty)$, depending only on q (and G), such that $\theta^b = 0$ if $\lambda/\delta < \rho_c$ and $\theta^b > 0$ if $\lambda/\delta > \rho_c$. We call ρ_c the* critical value *of the random-cluster model on $G \times \mathbb{R}$.*

The relevance of the space-time random-cluster measures to the quantum Ising (or more generally quantum Potts) model is explained in [3]; in particular *the ground state quantum Potts model on G exhibits long-range-order iff the corresponding random-cluster model has* $\theta^b > 0$. Hence, to investigate the phase-diagram of the quantum Ising model we will set $q = 2$ and focus on finding the critical value ρ_c above which percolation occurs.

Let us say a few more words about the "dual" $\mathbb{Y}$ of $\mathbb{X}$. Given any configuration $\omega \in \Omega$, one may associate with it a *dual* configuration on $\mathbb{Y}$ by placing a death wherever ω has a bridge, and a (hyper)bridge wherever ω has a death. This is illustrated in Fig. 4. More precisely, we let Ω_{d} be the set of pairs of locally finite subsets of $F \times \mathbb{R}$ and $W \times \mathbb{R}$, and for each $\omega = (B, D) \in \Omega$ we define its dual to be $\omega_{\mathrm{d}} := (D, B)$. As before, we may identify ω_{d} with its embedding in $\mathbb{Y}$, noting that some bridges may be embedded as polygons. We let Ψ_n^b and Ψ^b denote the laws of ω_{d} under Φ_n^{1-b} and Φ^{1-b} respectively.

The case when $G = \mathbb{Z}$ is particularly important, and for this case we use the lower case symbols ϕ and ψ in place of Φ and Ψ, respectively. When $G = \mathbb{Z}$, the dual space $\mathbb{Y}$ is isomorphic to $\mathbb{X}$, and we have the following result. Again the proof is similar to that for the discrete random-cluster model on $\mathbb{Z}^2$, but details for our model may be found in [6].

Lemma 1 *If ϕ_n^b, ϕ^b have parameters q, λ and δ, then the dual measures ψ_n^{1-b}, ψ^{1-b} are random cluster measures with parameters $q' = q$, $\lambda' = q\delta$ and $\delta' = \lambda/q$, and boundary condition $1-b$.*

Recall that there is a partial order on Ω given by $(B', D') = \omega' \geq \omega = (B, D)$ if $B' \supseteq B$ and $D' \subseteq D$, and that an event A is called *increasing* if whenever $\omega \in A$ and $\omega' \geq \omega$ then also $\omega' \in A$. Also recall that A is called a *cylinder event* if it only depends on a bounded region of $\mathbb{X}$, which is to say that there is n such that if $\omega = \omega'$ on Λ_n then $\omega \in A$ if and only if $\omega' \in A$.

Definition 3 Let κ be a probability measure on Ω.

- We say that κ is *positively associated* if for A, B any increasing cylinder events, $\kappa(A \cap B) \geq \kappa(A)\kappa(B)$.
- Another probability measure κ_1 on Ω *stochastically dominates* κ if for all increasing cylinder events A, we have $\kappa_1(A) \geq \kappa(A)$. We write $\kappa_1 \geq \kappa$.
- We say that κ has the *positivity property* if for all $\varepsilon > 0$ there exists a constant $0 < c = c(\varepsilon) < 1$ such that for all $e \in E, v \in V, t \in \mathbb{R}$,

$$c < \kappa(\text{no bridges in } \{e\} \times [t, t+\varepsilon]) < 1 - c \tag{8}$$

and

$$c < \kappa(\text{no deaths in } \{v\} \times [t, t+\varepsilon]) < 1 - c. \tag{9}$$

Proposition 2 *Let $q \geq 1$. The measures $\Phi_n^b, \Phi^b, \Psi_n^b, \Psi^b$ $(b = 0, 1)$ are positively associated and have the positivity property. Moreover, $\Phi^1 \geq \Phi^0$ and $\Psi^1 \geq \Psi^0$.*

The statement of Proposition 2 appears in [3] as well as in [1]; the proof is similar to the discrete random-cluster model and may be found in [6].

3 The Critical Value

We assume henceforth that $q = 2$. It is known that, if $G = \mathbb{Z}$, the critical value $\rho_c(2) = 2$. The following is the main result of this paper.

Theorem 2 *Let G be any star-like graph. Then the critical value on $G \times \mathbb{R}$ is $\rho_c(2) = 2$.*

In other words, the critical value for any star-like graph is the same as for $\mathbb{Z}$. Simpler arguments than those presented here can be used to establish the analogous result when $q = 1$, namely that $\rho_c(1) = 1$. Also, the same arguments can be used to calculate the critical probability of the discrete graphs $G \times \mathbb{Z}$ when $q = 1, 2$.

Here is a brief outline of the proof of Theorem 2. First we make the straightforward observation that $\rho_c(2) \leq 2$. Next, we use exponential decay to establish the existence of certain infinite paths in the dual model when $\lambda/\delta < 2$. Finally, we show how to put these paths together to form "blocking circuits" in $\mathbb{Y}$, which prevent the existence of infinite paths in $\mathbb{X}$ when $\lambda/\delta < 2$. Parts of the argument are inspired by [11].

Lemma 2 *For G any star-like graph, $\rho_c(2) \leq 2$.*

Proof Any star-like graph G contains an isomorphic copy of $\mathbb{Z}$ as a subgraph. Let Z be such a subgraph; we may assume that $\mathcal{O} \in Z$. Also we let ϕ_n^b, ϕ^b denote the random-cluster measures on $Z \times \mathbb{R}$. For each $n \geq 1$, let C_n be the event that no two points in $\Lambda_n \cap (Z \times \mathbb{R})$

are connected by a path which leaves $Z \times \mathbb{R}$. Clearly each C_n is a decreasing event. It follows from a standard property of random-cluster measures, sometimes called the DLR-property, that $\Phi_n^b(\cdot \mid C_n) = \phi_n^b(\cdot)$. The proof of this uses standard techniques [12, Theorem 3.7]; details for this model may be found in [6]. If A is an increasing cylinder event, this means that

$$\phi_n^b(A) = \Phi_n^b(A \mid C_n) \leq \Phi_n^b(A), \tag{10}$$

i.e. $\phi_n^b \leq \Phi_n^b$ for all n. Letting $n \to \infty$ it follows that $\phi^b \leq \Phi^b$. If $\lambda/\delta > 2$ then $\phi^b((\mathcal{O}, 0) \leftrightarrow \infty) > 0$ so then also

$$\Phi^b((\mathcal{O}, 0) \leftrightarrow \infty) > 0, \tag{11}$$

which is to say that $\rho_c(2) \leq 2$. □

3.1 Infinite Paths in the Half-Plane

Let us now establish some facts about the random-cluster model on $\mathbb{Z}_+ \times \mathbb{R}$ which will be useful later. Our notation is as follows: for $n \geq 1$,

$$\begin{aligned} S_n &= \{(a, t) \in \mathbb{Z} \times \mathbb{R} : -n \leq a \leq n, |t| \leq n\} \\ S_n(m, s) &= S_n + (m, s) = \{(a + m, t + s) \in \mathbb{Z} \times \mathbb{R} : (a, t) \in S_n\}. \end{aligned} \tag{12}$$

For brevity write $T_n = S_n(n, 0)$; also let ∂ denote the boundary,

$$\partial S_n = \{(a, t) \in \mathbb{Z} \times \mathbb{R} : a = \pm n \text{ or } t = \pm n\} \tag{13}$$

and $\partial S_n(m, s) = \partial S_n + (m, s)$. For $b = 0, 1$ and Δ one of S_n, T_n, we let ϕ_Δ^b denote the $q = 2$ random-cluster measure on Δ with boundary condition b and parameters λ, δ. Note that

$$\phi^b = \lim_{n \to \infty} \phi_{S_n}^b, \qquad \psi^b = \lim_{n \to \infty} \psi_{S_n}^b. \tag{14}$$

We will also be using the limits

$$\phi^w = \lim_{n \to \infty} \phi_{T_n}^1, \qquad \psi^f = \lim_{n \to \infty} \psi_{T_n}^0. \tag{15}$$

These are measures on configurations ω on $\mathbb{Z}_+ \times \mathbb{R}$; but according to our definition they cannot be random-cluster measures since the regions T_n do not tend to the whole of $\mathbb{Z} \times \mathbb{R}$. However, standard arguments let us deduce all the properties of ϕ^w, ψ^f that we need. In particular ψ^f and ϕ^w are mutually dual (with the obvious interpretation of duality) and they enjoy the positive association and positivity properties of Definition 3.

Let W be the "wedge"

$$W = \{(a, t) \in \mathbb{Z}_+ \times \mathbb{R} : 0 \leq t \leq a/2 + 1\}, \tag{16}$$

and write 0 for the origin $(0, 0)$.

Lemma 3 *Let $\lambda/\delta < 2$. Then*

$$\psi^f(0 \leftrightarrow \infty \text{ in } W) > 0. \tag{17}$$

Here is some intuition behind the proof of Lemma 3. The claim is well-known with ψ^0 in place of ψ^f, by standard arguments using duality and exponential decay. However, ψ^f is stochastically smaller than ψ^0, so we cannot deduce the result immediately. Instead we pass to the dual ϕ^w and establish directly a lack of blocking paths. The problem is the presence of the infinite "wired side"; we get the required fast decay of two-point functions by using the following result.

Proposition 3 *Let* $\lambda/\delta < 2$. *There is* $\alpha > 0$ *such that for all* n,

$$\phi^1_{S_n}(0 \leftrightarrow \partial S_n) \le e^{-\alpha n}. \tag{18}$$

In words, correlations decay exponentially under finite volume measures as soon as they do so under infinite volume measures. Results of this type for the classical Ising and random-cluster models appear in many places. In [8] and [9] it is proved for general $q \ge 1$ random-cluster models in two dimensions, and more general results about the two-dimensional case appear in [4]. A proof of general results of this type for the classical Ising model in any dimension appears in [14]. Below we adapt the argument in [14] to the current setting, with the difference that we shorten the proof by using the Lieb inequality in place of the GHS-inequality; use of the Lieb-inequality was suggested by Grimmett (personal communication). Note that the same argument works in any dimension.

Proof Let $\overline{S}_n \supseteq S_n$ denote the "tall" box

$$\overline{S}_n = \{(a,t) \in \mathbb{Z} \times \mathbb{R} : -n \le a \le n, |t| \le n+1\}. \tag{19}$$

We will use a variant of the random-cluster measure on $\overline{S}_n$ which has non-constant intensities for bridges and deaths, and also a process of *ghost-bonds*. To this end we create a new site g, which we think of as a "point at infinity", and let $\delta(\cdot), \gamma(\cdot) : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ and $\lambda(\cdot) : (\mathbb{Z} + 1/2) \times \mathbb{R} \to \mathbb{R}$ be bounded, nonnegative and measurable functions. Given independent Poisson processes of bridges and deaths of rates $\lambda(\cdot)$ and $\delta(\cdot)$, respectively, and of links to g of rate $\gamma(\cdot)$, we may define random-cluster measures as in Definition 2, where now any components connected to g are to be counted as the same.

The particular intensities we use are these. Fix n, and fix $m \ge 0$, which we think of as large. Let $\lambda(\cdot)$, $\delta(\cdot)$ and $\gamma_m(\cdot)$ be given by

$$\begin{aligned}
\delta(a,t) &= \begin{cases} \delta, & \text{if } (a,t) \in S_n \\ 0, & \text{otherwise,} \end{cases} \\
\lambda(a+1/2,t) &= \begin{cases} \lambda, & \text{if } (a,t) \in S_n \text{ and } (a+1,t) \in S_n \\ 0, & \text{otherwise,} \end{cases} \\
\gamma_m(a,t) &= \begin{cases} \lambda, & \text{if exactly one of } (a,t) \text{ and } (a+1,t) \text{ is in } S_n \\ m, & \text{if } (a,t) \in \overline{S}_n \setminus S_n \\ 0, & \text{otherwise.} \end{cases}
\end{aligned} \tag{20}$$

In words, the intensities are as usual "inside" S_n and in particular there is no external field in the interior; on the left and right sides of S_n, the external field simulates the wired boundary condition; and on top and bottom, the external field simulates an approximate wired boundary (as $m \to \infty$). We introduce another parameter $r \in [0,1]$, and let $\tilde{\phi}^r_{m,n}$ denote the random-cluster measure on $\overline{S}_n$ with intensities $\lambda(\cdot), \delta(\cdot), r\gamma_m(\cdot)$. Note that $\tilde{\phi}^0_{m,n}$ and $\phi^0_{S_n}$ agree on events defined on S_n, for any m.

Let X denote $\overline{S}_n \setminus S_n$ together with the left and right sides of S_n. By the Lieb inequality, proved for the space-time Ising formulation of the present model in [7] (see also [6]), we have that

$$\tilde{\phi}^1_{m,n}(0 \leftrightarrow g) \le e^{8\delta} \int_X dx\, \tilde{\phi}^0_{m,n}(0 \leftrightarrow x) \tilde{\phi}^1_{m,n}(x \leftrightarrow g) \le e^{8\delta} \int_X dx\, \tilde{\phi}^0_{m,n}(0 \leftrightarrow x), \tag{21}$$

since X separates 0 from g. Therefore, by stochastic domination by the infinite-volume measure,

$$\tilde{\phi}^1_{m,n}(0 \leftrightarrow g) \le e^{8\delta} \int_X dx\, \phi^0(0 \leftrightarrow x). \tag{22}$$

All the points $x \in X$ are at distance at least n from the origin. By exponential decay in the infinite volume, as proved in [7] using similar methods to the discrete case [2], there is an absolute constant $\tilde{\alpha} > 0$ such that

$$\tilde{\phi}^1_{m,n}(0 \leftrightarrow g) \le e^{8\delta} |X| e^{-\tilde{\alpha} n} = e^{8\delta}(8n+2)e^{-\tilde{\alpha} n}. \tag{23}$$

Now let C be the event that all of $\overline{S}_n \setminus S_n$ belongs to the connected component of g, which is to say that all points on $\overline{S}_n \setminus S_n$ are linked to g. Then by the DLR-property of random-cluster measures the conditional measure $\tilde{\phi}^1_{m,n}(\cdot \mid C)$ agrees with $\phi^1_{S_n}(\cdot)$ on events defined on S_n. Therefore

$$\begin{aligned} \phi^1_{S_n}(0 \leftrightarrow \partial S_n) &= \tilde{\phi}^1_{m,n}(0 \leftrightarrow \partial S_n \mid C) = \tilde{\phi}^1_{m,n}(0 \leftrightarrow g \mid C) \\ &\le \frac{\tilde{\phi}^1_{m,n}(0 \leftrightarrow g)}{\tilde{\phi}^1_{m,n}(C)} \le \frac{e^{8\delta}}{\tilde{\phi}^1_{m,n}(C)} \cdot (8n+2)e^{-\tilde{\alpha} n}. \end{aligned} \tag{24}$$

Since $\tilde{\phi}^1_{m,n}(C) \to 1$ as $m \to \infty$ we conclude that

$$\phi^1_{S_n}(0 \leftrightarrow \partial S_n) \le e^{8\delta}(8n+2)e^{-\tilde{\alpha} n}. \tag{25}$$

Since each $\phi^1_{S_n}(0 \leftrightarrow \partial S_n) < 1$ it is a simple matter to tidy this up to get the result claimed. □

Proof of Lemma 3 Let $T = \{(a, a/2+1) : a \in \mathbb{Z}_+\}$ be the "top" of the wedge W. We claim that

$$\sum_{n \ge 1} \phi^w((n,0) \leftrightarrow T \text{ in } W) < \infty. \tag{26}$$

Once this is proved, it follows from the Borel–Cantelli lemma that with probability one under ϕ^w, at most finitely many of the points $(n,0)$ are connected to T inside W. Hence under the dual measure ψ^f there is an infinite path inside W with probability one, and by the positivity- and positive association properties it follows that

$$\psi^f(0 \leftrightarrow \infty \text{ in } W) > 0, \tag{27}$$

as required.

To prove the claim we note that, if n is larger than some constant, then the event "$(n,0) \leftrightarrow T$ in W" implies the event "$(n,0) \leftrightarrow \partial S_{n/3}(n,0)$". The latter event, being increasing, is more likely under the measure $\phi^1_{S_{n/3}(n,0)}$ than under ϕ^w. But by Proposition 3,

$$\phi^1_{S_{n/3}(n,0)}((n,0) \leftrightarrow \partial S_{n/3}(n,0)) = \phi^1_{S_{n/3}}(0 \leftrightarrow \partial S_{n/3}) \le e^{-\alpha n/3}, \tag{28}$$

which is clearly summable. □

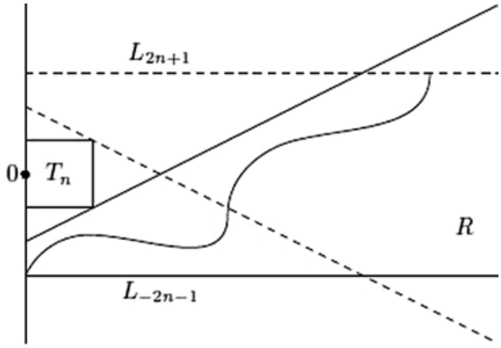

Fig. 5 Construction of a "half-circuit" in $\mathbb{Z}_+ \times \mathbb{R}$. With probability one, any infinite path in the lower wedge must reach the line L_{2n+1}, and similarly for any infinite path in the upside-down wedge. Any pair of such paths starting on the horizontal axis must cross

3.2 Proof of the Main Result

We prove one more lemma about the half-plane before going on to the main result; the proof uses a variant of standard blocking arguments.

Lemma 4 *Let* $\lambda/\delta < 2$. *There exists* $\varepsilon > 0$ *such that for each* n,

$$\psi^f((0, 2n+1) \leftrightarrow (0, -2n-1) \text{ off } T_n) \geq \varepsilon. \tag{29}$$

Proof Let $L_n = \{(a, n) : a \geq 0)\}$ be the horizontal line at height n, and let $\varepsilon > 0$ be such that $\psi^f(0 \leftrightarrow \infty \text{ in } W) \geq \sqrt{\varepsilon}$. We claim that

$$\psi^f((0, -2n-1) \leftrightarrow L_{2n+1} \text{ off } T_n) \geq \sqrt{\varepsilon}. \tag{30}$$

Clearly ψ^f is invariant under reflection in the x-axis, and standard arguments [12, Theorem 4.19] imply that it is also invariant under vertical translation. Thus once the claim is proved we get that

$$\begin{aligned} &\psi^f((0, 2n+1) \leftrightarrow (0, -2n-1) \text{ off } T_n) \\ &\quad \geq \psi^f((0, -2n-1) \leftrightarrow L_{2n+1} \text{ off } T_n \text{ and } (0, 2n+1) \leftrightarrow L_{-2n-1} \text{ off } T_n) \\ &\quad \geq (\sqrt{\varepsilon})^2, \end{aligned} \tag{31}$$

as required. See Fig. 5.

The claim follows if we prove that

$$\psi^f(0 \leftrightarrow \infty \text{ in } R) = 0, \tag{32}$$

where R is the strip

$$R = \{(a, t) : a \geq 0, -2n-1 \leq t \leq 2n+1\}. \tag{33}$$

However, (32) follows from the positivity property of Definition 3 and the Borel–Cantelli lemma, since the event "no bridges between $\{k\} \times [-2n-1, 2n+1]$ and $\{k+1\} \times [-2n-1, 2n+1]$" must happen for infinitely many k with ψ^f-probability one. To see this we can compare ψ^f with an independent percolation measure, as in the proof of Proposition 3. We have that $\psi^f \leq \mu$, where μ has parameters λ, δ; under μ the events above are independent, so

$$\psi^f(0 \leftrightarrow \infty \text{ in } R) \leq \mu(0 \leftrightarrow \infty \text{ in } R) = 0. \tag{34}$$

□

Proof of Theorem 2 We may assume that $G \neq \mathbb{Z}$, since the case $G = \mathbb{Z}$ is known. Let $\lambda/\delta < 2$, and recall that G consists of finitely many infinite "arms", where each vertex has degree two, together with a "central" collection of other vertices. On each of the arms, let us fix one arbitrary vertex (of degree two) and call it an *exit point*. Let U denote the set of exit points of G.

Given an exit point $u \in U$, call its two neighbours v and w; we may assume that they are labelled so that only v can reach the origin $\mathcal{O}$ without passing u. If the edge uv were removed from G, the resulting graph would consist of two components, where we denote by J_u the component containing w. Let $\hat{\Phi}_n^b$, $\hat{\Phi}^b$ denote the marginals of Φ_n^b, Φ^b on $X_u := J_u \times \mathbb{R}$; similarly let $\hat{\Psi}_n^b$, $\hat{\Psi}^b$ denote the marginals of the dual measures. Of course X_u is isomorphic to the half-plane graph considered in the previous subsection. By positive association and the DLR-property of random-cluster measures, $\hat{\Phi}_n^0 \leq \phi^1_{T_n(u)}$, so letting $n \to \infty$ also $\hat{\Phi}^0 \leq \phi^w$. Passing to the dual, it follows that $\hat{\Psi}^1 \geq \psi^f$. The (primal) edge uv is a *vertex* in the line-hypergraph; denoting it still by uv we therefore have by Lemma 4 that there is an $\varepsilon > 0$ such that for all n,

$$\Psi^1((uv, -2n-1) \leftrightarrow (uv, 2n+1) \text{ off } T_n(u) \text{ in } X_u) \geq \varepsilon. \tag{35}$$

Here $T_n(u)$ denotes the copy of the box T_n contained in X_u. Letting A denote the intersection of the events above over all exit points u, and letting $A_1 = A_1(n)$ be the dual event $A_1 = \{\omega_{\mathrm{d}} : \omega \in A\}$, it follows from positive association that $\Phi^0(A_1) \geq \varepsilon^k$, where $k = |U|$ is the number of exit points. Note that A_1 is a decreasing event in the primal model. The intuition is that on A_1, no point in $T_n(u)$ can reach ∞ without passing the line $\{u\} \times [-2n-1, 2n+1]$, since there is a dual blocking path in X_u.

Next let I denote the (finite) subgraph of G spanned by the complement of all the J_u for $u \in U$, and let $A_2 = A_2(n)$ denote the event that for all vertices $v \in I$, the intervals $\{v\} \times [2n+1, 2n+2]$ and $\{v\} \times [-2n-1, -2n-2]$ all contain at least one death and the endpoints of no bridges (in the primal model). By the positivity property, there is $\eta > 0$ independent of n such that $\Phi^0(A_2) \geq \eta$. So by positive association $\Phi^0(A_1 \cap A_2) \geq \eta\varepsilon^k > 0$. On the event $A_1 \cap A_2$, no point inside the union of $I \times [-n, n]$ with $\bigcup_{u \in U} T_n(u)$ can lie on an infinite path. See Fig. 6. Taking the intersection of the $A_1(n) \cap A_2(n)$ over all n, it follows that

$$\Phi^0(\text{there is no unbounded connected component}) \geq \eta\varepsilon^k. \tag{36}$$

The event that there is no unbounded connected component is a tail event. All infinite-volume random-cluster measures are tail-trivial (see [12, Theorem 4.19] or [6]), so it follows, whenever $\lambda/\delta < 2$, that

$$\Phi^0(0 \not\leftrightarrow \infty) = 1. \tag{37}$$

In other words, $\rho_c(2) \geq 2$. Combined with the opposite bound in Lemma 2, this gives the result. □

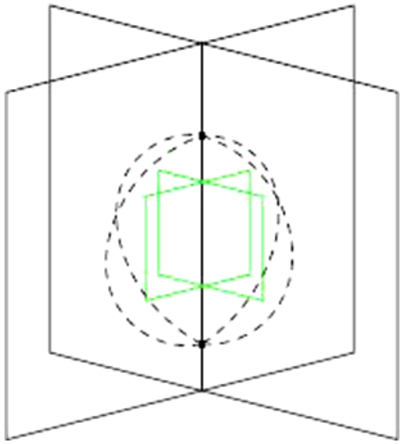

Fig. 6 The *dashed lines* indicate dual paths that block any primal connection from the interior to ∞. Note that this figure illustrates only the simplest case when G is a junction of lines at a single point

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